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Toward an Axiomatic Pregeometry of Space-Time

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We present a deductive theory of space-time which is realistic, objective, and relational. It is realistic because it assumes the existence of physical things endowed with concrete properties. It is objective because it can be formulated without any reference to knowing subjects or sensorial fields. Finally, it is relational because it assumes that space-time is not a thing, but a complex of relations among things. In this way, the original program of Leibniz is consummated, in the sense that space is ultimately an order of coexistents, and time is an order of successives. In this context, we show that the metric and topological properties of Minkowskian space-time are reduced to relational properties of concrete things. We also sketch how our theory can be extended to encompass a Riemannian space-time.

1. INTRODUCTION

Space-time is a primitive (*i.e.*, nonderivable) concept in every physical theory. Even so-called space-time theories like general relativity do not deal with the nature of space-time, but with its geometrical structure. The question "What is space-time?" precedes the formulation of any specific physical theory, and belongs to "protophysics" (*i.e.*, the branch of scientific ontology concerned with the basic assumptions of physics).

The ontological status of space-time has been a particular subject of debate for physicists and philosophers during the last 400 years. The kernel of this debate has been the confrontation of two antagonic positions: absolutism and relationalism. The former considers space-time as much a thing as planets and electrons, *i.e.*, space-time would be a physical entity endowed with concrete properties. This is the position held by Newton in his renowned

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discussion with Leibniz (mediated by S. Clarke; see Alexander, 1983), and also by J. Wheeler in the geometrodynamical approach to physics (*e.g.*, Misner *et al.*, 1973). Relationalism instead asserts that space-time is not a thing, but a complex of relations among physical things. In Leibniz's words: "I have said more than once, that I hold space to be something merely relative, as time is; that I hold it to be an order of coexistents, as time is an order of successions" (Alexander, 1983).

An important consequence of Leibniz's ideas is that, if space-time is not an ontological primitive, then it should be possible to construct it starting from a deeper ontological level. That is, the spatiotemporal relations should be definable from more fundamental relations. There have been several attempts to demonstrate the relational nature of space-time, both subjectivistic and phenomelogical (*e.g.*, Carnap, 1928; Basri, 1966) and objective and realistic (Bunge and Garcıa Maynez, 1977; Bunge, 1977). We think that a deductive theory of space-time cannot be built with blocks that are alien to physical discourse (such as knowing subjects or sensorial fields) in order to be compatible with contemporary physical theories. In this sense, we agree with Bunge's approach, which only assumes presuppositions common to the entire physical science (Bunge, 1977).

We present here a new formulation, realistic and objective, of the relational theory of space-time, based on the scientific ontology of Bunge (1977, 1979). The theory will be displayed as an axiomatic system, in such a way that its structure will turn out to be easily analyzable.[4] The construction of the theory rests on the notion of interaction among basic things, and on the notion of simultaneity.

At this point, we should mention that the search for a quantum theory of gravity has triggered intense research on the nature of space-time (for an exhaustive review, see Gibbs, 1996). The aim of this research is to build a theory ("pregeometry") from which all the "properties" of space-time (like continuity and dimensionality) can be explained.[5] This kind of pregeometry should be the consequence of the unavoidable merging of quantum mechanics and general relativity at very small distances. We emphasize that the pregeometry we propose here is valid only for lengths above a minimum length, which is suggested to be the Planck length by arguments based on the (yet unborn) theory of quantum gravity (Garay, 1995).

The structure of the paper is as follows: in Section 2 we offer a brief account of the main ontological assumptions of the theory. Section 3 contains some formal tools, such as uniform spaces, to be used later. The axiomatic

[4] On the advantages of the axiomatic method see Perez Bergliaffa *et al.* (1993) and references therein.

[5] Recall that, according to our view, space-time is not a thing. Consequently, it cannot have properties.

core is presented in Section 4. Finally, in Section 5 we give a short sketch of an extension of these ideas to Riemannian space-times, we compare our theory with the theory of Bunge (1977), and we close with some observations on the nature of space-time.

2. ONTOLOGICAL BACKGROUND

In this section we give a brief synopsis of the ontological presuppositions that we take for granted in our theory. For details see Bunge (1977, 1979) and Perez-Bergliaffa *et al.* (1996). The basic statements of the ontology can be formulated as follows:

1. There exist concrete objects named *things*. The set of all things is denoted by Θ.

2. Things can juxtapose ($\dot{+}$) and superimpose ($\dot{\times}$) to give new things according to the following definitions:

(a) A thing x is a physical sum or juxtaposition (denoted by $+$) of all the individuals of a given set $\{x_i\}$ iff every part of x is a part of at least one of the members of the set. Example: the juxtaposition of an electron and a proton yields a hydrogen atom.

(b) A thing x is a physical product or superposition (denoted by $\times$) of all the individuals of a given set $\{x_i\}$ iff every part of x is a part of every member of the set. Example: the superposition of two electromagnetic fields yields another electromagnetic field.

3. The null thing $\Diamond$ is a fiction introduced in order to give the structure of Boolean algebra to the laws of composition of things:

$$x \dot{+} \Diamond = x$$

$$x \dot{\times} \Diamond = \Diamond$$

4. Two things are separated if they do not superimpose:

$$x \,\text{—}\, y \Leftrightarrow x \dot{\times} y = \Diamond$$

5. Let T be a set of things. The *aggregation* of T (denoted $[T]$) is the supremum of T with respect to the operation $\dot{+}$.

6. The world ($\Box$) is the aggregation of all things:

$$\Box = [\Theta] \Leftrightarrow (x \sqsubset \Box \Leftrightarrow x \in \Theta)$$

where the symbol $\sqsubset$ means 'is part of.' It stands for a relation between concrete things and should be not mistaken with '$\in$', which is a relation between elements and sets (*i.e.*, abstract entities).

7. All things are composed of basic things $x \in \Xi \subset \Theta$ by means of juxtaposition or superimposition. The basic things are elementary or primitive:

$$(x, y \in \Xi) \wedge (x \sqsubset y) \Rightarrow x = y$$

8. All things have *properties P.* These properties can be intrinsic or relational.

9. The *state* of a thing is a set of functions from a domain of reference M to the set of properties $\mathcal{P}$. The set of accessible states of a thing x is the *lawful state space* of x: $S_L(x)$. The state of a thing is represented by a point in $S_L(x)$.

10. A *law statement* is a restriction upon the state functions of a given class of things. A *natural law* is a property represented by an empirically corroborated legal statement.

11. The *history* $h(x)$ of a thing x is the part of $S_L(x)$ defined by

$$h(x) = \{\langle t, F(t)\rangle | t \in M\}$$

where t is an element of some auxiliary set M, and F are the functions that represent the properties of x.

12. Two things *interact* if each of them modifies the history of the other:

$$x \bowtie y \Leftrightarrow h(x + y) \neq h(x) \cup h(y)$$

13. A thing x_f is a *reference frame* for x iff (i) M equals the state space of x_f, and (ii) $h(x \dot{+} f) = h(x) \cup h(f)$

14. A *change* of a thing x is an ordered pair of states:

$$(s_1, s_2) \in E_L(x) = S_L(x) \times S_L(x)$$

A change is called an *event*, and the space $E_L(x)$ is called the *event space* of x.

15. An event e_1 *precedes* another event e_2 if they compose to give $e_3 \in E_L(x)$:

$$e_1 = (s_1, s_2) \wedge e_2 = (s_2, s_3) \Rightarrow e_3 = (s_1, s_3)$$

The ontology sketched here (due mainly to M. Bunge) is realistic, because it assumes the existence of things endowed with properties, and objective, because it is free of any reference to knowing subjects.

We will base the axiomatic formulation of the pregeometry of space-time on this ontology and on the formal tools that will be described in the following.

3. FORMAL TOOLS

3.1. Topological Spaces

We give here just a brief review; for details the reader is referred to Thron (1966) and references therein.

D1. $\mathcal{P}(A) =_{\mathrm{Df}} \{X/X \subset A\}$ is the *power set* of the set A.

D2. Let A be a set. A subset $\mathfrak{X}$ of $\mathcal{P}$(A) is a *topology* on A if:

1. $\emptyset \in \mathfrak{X}$, $A \in \mathfrak{X}$
2. If $A_i \in \mathfrak{X}$, $i \in [i_1, \ldots, i_n]$, then $\cup_{i=1}^{n} A_i \in \mathfrak{X}$
3. If $A_i \in \mathfrak{X}$, $i \in [i_1, \ldots, i_n]$, then $\cap_{i=1}^{n} A_i \in \mathfrak{X}$

The elements of $\mathfrak{X}$ are usually known as the *open sets* of A. The pair $(A, \mathfrak{X})$ is called a *topological space*. The elements of A on which a topology $\mathfrak{X}$ is defined are the *points* of the space $(A, \mathfrak{X})$.

D3. A family $\mathcal{B} \in \mathcal{P}$(A) is a *base* iff the family $\mathfrak{X}$ of all unions of elements of $\mathcal{B}$ is a topology on $\cup\{B/B \in \mathcal{B}\}$. It is said then that $\mathfrak{X}$ is the topology *generated* by $\mathcal{B}$.

3.2. Filters

D4. A nonempty family $\mathcal{B}$ of subsets of a set A is a *filter* on A iff:

1. $(A_1 \in \mathcal{F} \wedge A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F})$
2. $B \supset A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$
3. $\emptyset \notin \mathcal{F}$

D5. A nonempty family B of subsets of a set A is called a *filter base* on A provided B does not contain the empty set and provided the intersection of any two elements of B contains an element of B

3.3. Uniform Spaces

D6. A nonvoid family Λ of subsets of $A \times A$ is a *uniformity* on A iff

1. $L \supset \triangle$, where $\triangle = \{(x, y)/x \in A \wedge y \in A \wedge x = y\}$ for all $L \in \Lambda$
2. $C \supset L \in \Lambda$ implies $C \in \Lambda$
3. $L_1, L_2 \in \Lambda \Rightarrow L_1 \cap L_2 \in \Lambda$
4. $L \in \Lambda \Rightarrow L^{-1}$, where $L^{-1} = \{(x, y)/(y, x) \in L\} \in \Lambda$
5. For all $L \in \Lambda$ there exists a $K \in \Lambda$ such that $K \circ K \subset L$, where $K \circ K = \{(x, y)/\exists z[(x, z) \in K \wedge (z, y) \in K\}$

D7. The pair (A, Λ) is called a *uniform space*.

D8. (A, Λ) is called a separated (or Hausdorff) uniformity iff $\cap\ L \in \Lambda = \Delta$

Remark. Notice that a uniformity is a filter on $A \times A$ each element of which contains Δ. Property 4 is a symmetry property, whereas property 5 is an abstract version of the triangle inequality.

D9. A set B is called *everywhere dense* in a set A iff $\overline{B}$ (the closure of B) $\supset A$.

D10. A topological space (X, τ) is *separable* iff there exists an everywhere dense subset of X which is denumerable.

D11. A filter $\mathcal{F}$ in a uniform space (A, Λ) is called a *Cauchy filter* iff, given $L \in \Lambda$, there exists an $M \in \mathcal{F}$ such that $M \times M \subset L$. A uniform space is *complete* iff every Cauchy filter has a limiting point.

T1. Every separated uniform space has a *completion*. That is, one can always add "ideal elements" to complete the space.

Proof. See Thron (1966), pp. 184–185.

3.4. Metric Spaces

D12. Let X be a set. A function d: $X \times X \mapsto R^+$ is a *metric* on X iff:

1. $d(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$
2. $x = y \Rightarrow d(x, y) = 0$ for all $x, y \in X$
3. $d(x, y) = d(y, x)$ for all $x, y \in X$
4. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

D13. The pair (X, d) is a *metric space*.

T2 (Theorem of metrization). A uniform space is *metrizable* if and only if it is separable and its uniformity has a numerable base (Kelley, 1962).

T3 (Theorem of isometric completion). Any metric space is isometric to a subspace dense in a complete metric space (Kelley, 1962).

T4. Let S be a subset of X and let $(X, \mathcal{H})$ be a uniform space. Then the family $\mathcal{H}_S = \{H \cup (S \times S)/H \in \mathcal{H}\}$ is a uniformity on S (called the relativized uniformity), and $\tau_{\mathcal{H}_s} = (\tau_{\mathcal{H}})_S$.

4. AXIOMATICS

We present now the axiomatic core of our formulation. The generating basis of primitive concepts is

$$B = \{\Xi, \mathcal{P}, S_L, E_o, E_G, T_u, \dot{+}, \dot{\times}, \leq, c\}$$

The different symbols are characterized by the ontological background (Section 2) and a set of specific axioms. We shall classify these axioms into ontological (o), formal (f), and semantical (s), according to their status in the theory.

A1. (o) For each $x \in \Xi$ there exists a single ordering relation:

$$s_1 \leq s_2 \Leftrightarrow s_2 = g(s_1)$$

where g: $S_L \to S_L$ is a law statement.

A2. (s) The set of lawful states of x, $S_L(x)$, is (*temporally*) *ordered* by the relation $\leq$.

D14. $s_1 \leq s_2 \Leftrightarrow s_1$ *precedes (temporally)* s_2.

Remark. The relation $\leq$ is a partial order relation: there are states that are not ordered by $\leq$ (e.g., given the initial conditions x_0, v_0, there are states which are characterized by the values of x and v that cannot be reached by a classical particle).

D15. A subset of $S_L(x)$ totally ordered by the relation $\leq$ is called a *proper history of x*.

A3. (o) For each thing x, there exists one and only one proper history.

A4. (o) If the entire set of states of an ontological history is divided into two subsets h_p and h_f such that every state in h_p temporally precedes any state in h_f, then there exists one and only one state s_0 such that $s_1 \leq s_0 \leq s_2$, where $s_1 \in h_p$ and $s_2 \in h_f$. In symbols:

$$(\forall s_1)_{h_p}(\forall s_2)_{h_f}(s_1 \leq s_2)(\exists s_0)(s_1 \leq s_0 \leq s_2)$$

Remark. This axiom expresses the notion of *ontological continuity*.

D16. h_p is called the *past* of s_0, and h_f is called the *future* of s_0.

Remark. Notice that past and future are meaningful concepts just when they are referred to a given state s_0.

A5. (o) For every thing x, there exists another thing x_t called a *clock*, and an injective application ψ such that:

1. ψ_t: $S_L(x_t) \to S_L(x)$
2. Given t, $t' \in S_L\ (x_t)$: $t \leq t' \Rightarrow \psi(t) \leq \psi(t')$

T5. Given a thing x with ontological history $h\,(x)$ and an arbitrary system of units U_τ there exists a bijection

$$\mathcal{T}: \quad h \times U_\tau \leftrightarrow R$$

that gives a parametrization $s_x = s_x(\tau)$.

Proof. From **A3**, **A4**, and Rey Pastor *et al.* (1952).

D17. The variable τ is called the *proper time* of x.

T6. Let x_t be a clock for x, with event space $E_L(x_t)$, and U_t an arbitrary system of units. There exists a bijection

$$T: \quad E_L(x_t) \times U_t \leftrightarrow R$$

that provides a parametrization $s = s(\tau)$.

Proof. Generalization of **T5**.

D18. t is the *duration* of an event of x relative to the clock x_t.

This is all we need to say about time. For more details, see Bunge (1977).

A6:

$$(\forall x)(x \sqsubset \Box)(\exists y)(y \sqsubset \Box \wedge y \bowtie x)$$

Remark. This axiom states that there exist no completely isolated things.

We shall now show that the relation of interaction $\bowtie$ (see Section 2), generalized in a convenient way, induces a uniform structure (see Section 3) on the set of basic things. It is important to note that the relation $\bowtie$ is symmetric, but neither reflexive nor transitive. However, it is always possible to define a reflexive-transitive closure of a given relation (Salomaa, 1973). The closure $\bowtie^*$ of the relation of interaction is the set of pairs of basic things that interact either directly or by means of a chain (finite or infinite) of basic things.

Now the following theorem can be proved:

T7. The relation $\bowtie^*$ defines a uniform structure on Ξ.

Proof. Every equivalence relation defines a uniform structure on a set (Thron 1966).

Remark. Armed with this theorem, we will be able to endow space with a uniform structure.

In order to introduce the concept of space, we shall use the notion of *reflex action* between two things. Intuitively, a thing x acts on another thing y if the presence of x disturbs the history of y. Events in the real world seem to happen in such a way that it takes some time for the action of x to propagate up to y. This fact can be used to construct a relational theory of space à la Leibniz, that is, by taking space as a set of equitemporal things. It is necessary then to define the relation of simultaneity between states of things.

Let x and y be two things with histories $h(x_\tau)$ and $h(y_\tau)$, respectively, and let us suppose that the action of x on y starts at τ_x^0. The history of y will be modified starting from τ_y^0. The proper times are still not related, but we can introduce the reflex action to define the notion of simultaneity. The action of y on x, started at τ_y^0, will modify x from τ_x^1 on. The relation "the action of x is reflected on y and goes back to x" is the reflex action. Historically, Galilei (1945) introduced the reflection of a light pulse on a mirror to measure the speed of light. With this relation we will define the concept of simultaneity of events that happen in different basic things (see also Landau and Lifshitz 1967).

We have already seen in Section 2 that a thing x acts upon a thing y if the presence of x modifies the history of y:

$$x \rhd y =_{\mathrm{Df}} h(y|x) \neq h(y)$$

where $h(y|x)$ represents the history of y in the presence of x.

The *total action* of x upon y is

$$\mathscr{A}(x, y) = h(y|x) \cap \overline{h(y)}$$

where the bar designates the complement.

Let us now define the history of x after τ_x^0 as

$$h(x, \tau_x^0) = h(x)|_{\tau_x > \tau_x^0}$$

and similar definitions for $h(y, \tau_y^0)$ and for the history of y after τ_y^0 in the presence of x after τ_x^0, denoted here as $h(\langle y, \tau_y^0\rangle, \langle x, \tau_x^0\rangle)$.

The total action of x after τ_x^0 on y after τ_x^0 is

$$\mathscr{A}(y, x^0) = h(y|x) \cap \overline{h(\langle y, \tau_y^0\rangle, \langle x, \tau_x^0\rangle)}$$

In a similar way we define the action of y on x after τ_y^1.

τ_y^0 is the minimum value of the proper time of y for which the action of x after τ_x^0 is felt:

$$\tau_y^0 = \inf\{\tau_y | \mathscr{A}(y, x^0)\}$$

This quantity always exists, because of the ontological continuity assumed in **A4**.

Similarly, we define τ_x^1:

$$\tau_x^1 = \inf\{\tau_x | \mathscr{A}(x, y^0)\}$$

Finally we can introduce a relation between the three instants involved in the reflex action. We will call $\mathscr{R}\langle\tau_x^0, \tau_y^0, \tau_x^1\rangle$ the relation given by the set of ordered 3-tuples and established by the previous equations.

Let us go back to the axiomatics.

A7. (o) Given two different and separated basic things x and y, there exists a minimum positive bound for the interval $(\tau_x^1 - \tau_x^0)$, defined by $\mathscr{R}$.

Remark. Hereafter we shall deal only with 3-tuples $\langle\tau_x^0, \tau_y^0, \tau_x^1\rangle$ that satisfy the minimum condition.

D19. τ_y^0 is simultaneous with $\tau_x^{1/2} =_{\mathrm{Df}} (1/2)(\tau_x^0 + \tau_x^1)$.

T8. τ_x and τ_y can be *synchronized* by the simultaneity relation.

Proof. There exists a bijection between τ_x and τ_y because $\mathcal{R}^{-1}$, the inverse of $\mathcal{R}$, is well defined.

Comment. As we know from general relativity, the simultaneity relation is transitive only in special reference frames called *synchronous* (Landau and Lifshitz, 1967). We then include the following axiom:

A8. (f) Given a set of basic things $\{x_1, x_2, \ldots\}$, there exists an assignment of proper times $\tau_1, \tau_2, \ldots$ such that the relation of simultaneity is transitive.

T9. The relation of simultaneity is an equivalence relation.

Proof. From **T8** and **A7**.

Remark. We should mention that, because of **T5** and **D17**, the history of a given thing is parametrized by its proper time τ. Then, the relation of simultaneity is defined not over things, but over states of things.

D20. The equivalence class of states defined by the relation of simultaneity on the set of all basic things is the *ontic space* E_o.

T10. The ontic space E_o has a uniform structure.

Proof. Let S be a set of states of things related by the simultaneity relation. Because of the uniqueness of the ontological history postulated in **A3**, there is a one-to-one relation between a state in S and a given thing, and then S is isomorphic to a subset of Ξ. Then, by **T4**, S is a uniform space.

A9. There exists a subset D in the set of simultaneous states of interacting things S that is denumerable and dense in S.

Remark. This axiom requires space to be a *plenum*. Indeed, this hypothesis (introduced by Aristotle and later supported by Leibniz) is central to quantum physics, and it permits the prediction of a plethora of vacuum phenomena (such as the Casimir effect), in good agreement with observation.

A10. Each $x \in \Xi$ interacts with a denumerable set of basic things.

T11. The power set of Ξ reduced to the equivalence class is a basis for the uniformity (Bourbaki, 1964).

So now we have the necessary and sufficient conditions for the metrization theorem.

T12. The ontic space is metrizable.

Proof. Immediate, from **T2**.

D21. τ_u is the proper time of a reference thing x_f.

T13. $(\forall x) \ominus (\exists f_x)(\tau_x = f_x^{-1}(\tau_u))$

Proof. Immediate, from **A5**.

Remark. We shall call τ_u the *universal time*.

The ontic space E_0 is still devoid of any geometric properties and consequently cannot represent the physical space. We postulate then:

A11. (f) The metrization of the ontic space is given by

$$d(x, y) = \frac{1}{2}\, c|\tau_x^1 - \tau_x^0|$$

where c is a constant with appropriate dimensions, and the distance is evaluated at τ_y^0, which is simultaneous with $\tau_x^{1/2}$.

T14. The ontic space is isometric to a subspace dense in a complete space.

Proof. The proof follows immediately form the theorem of isometric completion (**T3**).

D22. The complete space mentioned in **T14** is called *geometric space E_G*.

Remark. Because of the isometry mentioned in **T14**, E_G inherits the metric of E_o. Besides, note that every filter of Cauchy has a limiting point in E_G, because this space is complete.

D23. The elements of the completion are called *ideal things*.

Remark. It should be noted that the ideal things (which are abstract objects) do not belong to the ontic space, but to the geometric space.

A12. (f) The points in E_G satisfy the following conditions:

1. Given two points x and y, there exists a third point y aligned with x and z.
2. There exist three nonaligned points.
3. There exist four noncoplanar points.
4. There exist only three spatial dimensions.[6]

Remark. All the conditions in **A12** can be expressed in terms of the distance $d(a, b)$. For instance, condition 1 can be written in the form

[6]This may not be true at the Planck scale, see, e.g., Tegmark (1997).

$$(\forall(a))_G(\forall(b))_G(\exists c)_G(d(a,\, b) + d(b,\, c) = d(a,\, c)\vee \tag{1}$$

$$d(a,\, c) + d(c,\, b) = d(a,\, b)\vee \tag{2}$$

$$d(c,\, a) + d(a,\, b) = d(a,\, c)) \tag{3}$$

For details, see Blumenthal (1965).

T15. The geometric space E_G is globally Euclidean.

Proof. From **A12**; see Blumenthal (1965).

Remark. The ontic space E_o is not Euclidean, but dense on a Euclidean space (*i.e.*, E_G). This is a consequence of the fact that a *sequence of Cauchy of things* does not have in general a thing as a limit.

D24. T_u is the the bijection alluded to in **T6** for the reference thing x_f with proper time τ_u.

T16. There exists a nontrivial geometric structure on $E_G \times T_u$.

Proof. Let us introduce a Cartesian coordinate system in E_G, with origin located in the reference thing x_f. From **T6**, **D19**, and **A11**,

$$(t_y - t_x)^2 = \left[\frac{d(y,\, x)}{c}\right]^2 \tag{4}$$

This equation describes a sphere of radius $c\ dt$ centered at x. Then the family of spheres $S(x, t_x) \subset E_G \times T_u$ defines a geometric structure on $E_G \times T_u$.

A13. (o) The *cones of action* determined by (4) are independent of the reference thing x_f.

T17. The quadratic form

$$ds^2 = (c\ dt)^2 - (d\bar{r})^2 \tag{5}$$

is invariant under changes of the reference thing.

Cor. 1. $E_G \times T_u$ has a Minkowskian structure.

T18. The only coordinate transformations that leave invariant the quadratic form (5) are the Lorentz transformations.

A14. (s) $E_G \times T_u$ represents physical space-time.

This axiom completes the formulation of the theory. It should be noted that the spatial relations that we perceive are defined between macroscopic (i. e., composed) things. Our system of axioms also can handle this situation (which strictly does not belong to protophysics, but to physics) if we incorporate a specific model for a given thing, which should be based on an explicit form of the interaction between basic things.

5. FURTHER COMMENTS

5.1. Extension to Quantum Basic Things

The relational theory of space-time expounded in the preceding section is based on the concept of basic thing. We should remark that this is a theory-dependent concept, i.e., two different theories may take different sets of things as basic. In this sense, our theory is classical (as opposed to quantum), because **T2** enforces the separability of basic things. However, it can be conveniently modified to serve as a part of the protophysics of quantum theories, and this we shall do in the following.

The main problem is the fact that quantum particles can superimpose (i.e., the distance between two of them can be zero while the particles are still distinguishable). To incorporate this fact, we shall relax **D12**, keeping only items 2–4. With this modification, **D12** defines a pseudometric instead of a metric (Kelley, 1962).

Now we replace **T2** with the following:

T19. (Theorem of pseudometrization). A uniform space is pseudometrizable if its uniformity has a denumerable base.

So, due to **A10**, E_o is pseudometrizable. Moreover, it can be completed:

T20. Every pseudometric space is isometric to a subspace dense in a complete pseudometric space.

We shall call this last space E_p (*pregeometric space*). But we know that in quantum mechanics the Euclidean space is included in the corresponding protophysics (Perez Bergliaffa *et al.*, 1993). To recover Euclidean space, we begin by introducing the concept of ontic point:

D25. Let $X \sqsubseteq \Xi$ be a family of basic things. We say that X is a *complete family of partially superimposed things* if

1. $(\forall x)_X (\forall y)_X (x \dot{\times}) y \neq \Diamond)$
2. $(\forall x)_{\not X} (\exists y)_X (x — y)$

We shall call this kind of family an *ontic point*, because the (pseudometric) distance between any two components of the family is zero.

A15. Let ξ and η be two ontic points. Then $(\forall x_i)_\xi (\forall y_j)_\eta (\exists C(\xi, \eta))(d_p(x_i, y_j) < C(\xi, \eta))$.

D26. Let ξ and η be two ontic points. The distance between them is given by

$$d_G(\xi, \eta) =_{\mathrm{Df}} \sup_{(i,j)} d_p(x_i, y_j) \tag{6}$$

with $x_i \in \xi$ and $y_j \in \eta$.

Remark. The axiom **A15** guarantees that this distance is well defined.

T21. The set of ontic points, together with the distance function (6), is a metric space.

Proof. The items 2–4 of **D12** are trivially satisfied because d_p is a pseudometric. Regarding the first item, if ξ and η are two ontic points, there exist $x_i \in \xi$ and $y_j \in \eta$ such that x_i — y_j. The condition $d_p(x_i, y_j) > 0$ is satisfied because d_p is a pseudometric. Then, $\xi \neq \eta \Rightarrow d_G(\xi, \eta) > 0$, which can be written as $d_G(\xi, \eta) = 0 \Rightarrow \xi = \eta$.

T22 (Theorem of isometric completion). Any metric space is isometric to a subspace dense in a complete metric space (Kelley, 1962).

D27. The *completion* of the space of ontic points is the geometric space E_G.

From this point, the construction goes on as in the previous case.

5.2. Extension to Riemannian Spaces

Our theory is a pregeometry for a Minkowskian space time. Gravitational physics, however, requires more complex structures. In this section we shall sketch the necessary steps that lead to a Riemanian space, which is used in general relativity (Covarrubias, 1993), in an informal way, avoiding the technicalities.

A Riemannian space can be obtained from our theory using a tetrad formulation. For this we need at least an axiom elucidating the connection between ontic and geometric spaces:

A16. (f) The geometric space E_{G} is the tangent space to the ontic space at the reference thing x_{f}.

With this axiom, the connection between ontic and geometrical spaces will be purely local. The full space will be constructed by pasting together patches of quasi-Euclidean pieces. The following axiom sketches the way this can be done:

A17. (f) There exists a parallel displacement operator connecting the components of vectors (i.e., elements of the tangent space) tangent to E_{o} on neighboring things.

With parallel displacement, a covariant derivative can be defined in the usual way. The usual property of the Riemannian conection (Ricci coefficients) must be posited. The following axiom will do the job:

A18. (f) The covariant derivative ∇ annihilates the metric.

Since we are working with a transitive simultaneity relation (which is equivalent to using a synchronous reference frame in any metric theory of gravitation; Landau and Lifchitz, 1967) a Riemannian space will define a unique pseudo-Riemannian spacetime. In this way a protophysics for a rigorous formulation of general relativity (such as Covarrubias, 1993) and more general theories of gravitation will be obtained.

The above scheme must be completed in several ways. Accurate definitions should be given of the different constructs defined. Also, several axioms should be introduced to ensure a differential manifold structure on a suitable completition of the ontic space. We shall not pursue further this matter here, but leave it to a future communication.

5.3. Comparison with Bunge's Theory

As mentioned in the Introduction, protophysical theories can be partitioned into subjective and objective, according to whether or not knowing subjects and/or sensorial fields are considered as basic objects. Bunge (1977) developed an objective and realistic relational theory of space time and made a clear-cut comparison with other subjective and objective theories. In this section, we shall limit ourselves to a comparison of the objective and realistic theory of Bunge (1977) with the one developed in the present paper.

Bunge's theory of space is based on the *interposition relation* $(x|y|z)$, which can be read "y interposes between x and z". The properties of this relation are posited in Axioms 6.1–6.6 of Bunge (1977). In this section we shall show how the corresponding relation can be constructed in the present theory.

We shall first define a similar relation between basic things.

D28. Let $x, y, z \in \Xi$. We shall say that $[x|y|z]_\Xi$ if

$$(d(x, y) + d(y, z) = d(x, z)) \wedge (x — y — z — x) \vee (x = y = z))$$

The next theorem proves that in our theory, the interposition relation holds between basic things if and only if it is valid in Bunge's theory:

T23. Let $x, y, z \in \Xi$. Then $[x|y|z]_\Xi$ if $(x|y|z)$.

Proof. The proof consists in showing that $[x|y|z]_\Xi$ satisfies each of the seven conditions (i)–(vii) of Axiom 6.1 in Bunge (1977).

In order to define an interposition relation for general things, we shall use our interposition relation for basic things:

D29. Let $\xi, \eta, \zeta \in \Theta$. Then $[\xi|\eta|\zeta]_\Theta$ either if they are equal ($\xi = \eta = \zeta$) or if there exist three separate basic things $x, y, z \in \Xi$ that are parts of one thing, but not of the others, and that interpose.

$$\begin{aligned}[\xi|\eta|\zeta]_\Theta =_{\mathrm{Df}} (\xi = \eta = \zeta)\vee \\ &\exists(x, y, z \in \Xi)\{[(x \sqsubset \xi) \wedge (x — \eta) \wedge (x — \zeta)] \wedge \\ &[(y \sqsubset \eta) \wedge (y — \xi) \wedge (y — \zeta)]\wedge \\ &[(z \sqsubset \zeta) \wedge (z — \eta) \wedge (z — \xi)] \wedge \\ &[x|y|z]_\Xi\} \end{aligned} \tag{7}$$

T24. Let $\xi, \eta, \zeta \in \Theta$. *Then* $[\xi|\eta|\zeta]_\Theta$ *if* $(\xi|\eta|\zeta)$.

In the same way, with appropriate definitions, it is possible to show that the remaining postulates of Bunge's theory can be recovered as theorems in our formulation.

The time theory exposed in the first part of our axiomatics is essentially the same theory exposed in Bunge (1977), although our axioms are somewhat different. The main differences lies in axiom 3 and 4. The first one, not explicitly stated in Bunge (1977), forbids "gardens of forking paths" (Borges, 1967), or, in general, more than one timelike direction. The second axiom may be taken as a reformulation of the Heraclitean principle, "*Panta rhei.*"

From a formal point of view, the present theory of space-time is very different from the theory in Bunge (1977). This is because our fundamental relation of "reciprocal action" is very restrictive and the related axioms are extremely strong: we are led almost without ambiguity to a Minkowskian structure of space-time.

Finally, it is important to remark that, since both theories have the same referents (namely, things and their properties), they are referentially equivalent, realistic and objective relational theories of space and time.

5.4. The Nature of Space-Time

In the present theory, space-time is not a thing, but a substantial property of the largest system of things, the world $\Box$, emerging from the set of the relational properties of basic things. Thus, any existential quantification over space-time can be translated into quantification over basic things. This shows that space-time has no ontological independence, but is the product of the interrelation between basic ontological building blocks. For instance, rather than stating "space-time possesses a metric," one should say, "the evolution of interacting things can be described attributing a metric tensor to their spatiotemporal relationships." In the present theory, however, space-time is interpreted in an strictly materialist and Leibnizian sense: it is an order of successive material coexistents.

6. SUMMARY

We have developed a materialist relational theory of space-time that carries out the program initiated by Leibniz and provides a protophysical basis consistent with any rigorous formulation of general relativity. Space-time is constructed from general concepts which are common to any consistent scientific theory. It is shown, consequently, that there is no need for positing the independent existence of space-time over the set of individual things.

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